AEDC-TR-79-35
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Nonorthogonal Measurement Axes
in Laser Doppler Velocimetry

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August 1979

Final Report for Period October 1977 — September 1978

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AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE

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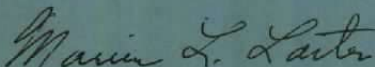
APPROVAL STATEMENT

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER AEDC-TR-79-35	2 GOVT ACCESSION NO	3 RECIPIENT'S CATALOG NUMBER
4 TITLE (and Subtitle) NONORTHOGONAL MEASUREMENT AXES IN LASER DOPPLER VELOCIMETRY		5 TYPE OF REPORT & PERIOD COVERED Final Report – October 1977 – September 1978
		6 PERFORMING ORG REPORT NUMBER
7 AUTHOR(s) F. L. Crosswy, ARO, Inc., a Sverdrup Corporation Company		8 CONTRACT OR GRANT NUMBER(s)
9 PERFORMING ORGANIZATION NAME AND ADDRESS Arnold Engineering Development Center/DOTR Air Force Systems Command Arnold Air Force Station, Tennessee 37389		10 PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element 65807F
11 CONTROLLING OFFICE NAME AND ADDRESS Arnold Engineering Development Center/OIS Air Force Systems Command Arnold Air Force Station, Tennessee 37389		12 REPORT DATE August 1979
		13 NUMBER OF PAGES 26
14 MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15 SECURITY CLASS (of this report) UNCLASSIFIED
		15a DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16 DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17 DISTRIBUTION STATEMENT (of the abstract entered in Block 20, If different from Report)

18 SUPPLEMENTARY NOTES

Available in DDC

19 KEY WORDS (Continue on reverse side If necessary and Identify by block number)

Doppler systems
laser velocimeters
fluid flow
measurement
orthogonality
Cartesian coordinates
information
velocity

20 ABSTRACT (Continue on reverse side If necessary and Identify by block number)

The measurement axes of virtually all multiaxis laser Doppler velocimeter (LDV) systems are nonorthogonal to some degree and are, in general, rotated with respect to orthogonal, Cartesian reference axes. Systematic errors are introduced into the LDV data if axis nonorthogonality and spatial orientation parameters are not properly taken into account by the data reduction equations. This report develops the general 3-space expressions for velocity vector

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20. ABSTRACT (Continued)

magnitude and spatial orientation in terms of rotated, oblique
LDV measurement axis parameters and the velocity vector
projections measured by each of these axes.

PREFACE

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ILLUSTRATIONS

1.0 INTRODUCTION

A series of optical homodyne experiments (Refs. 1 through 4) during the period 1962-1964 demonstrated the feasibility of exploiting the Doppler shift of scattered laser light for measuring the velocity of moving matter. Of particular importance to the aerodynamicist was the demonstration by Yeh and Cummins in 1964 (Ref. 4) that fluid velocities could be inferred by measuring the Doppler shift frequency of laser light scattered by particulate matter entrained in the fluid.

Since 1964, worldwide interest. intensive development efforts, and increasingly refined applications have resulted in significant improvements in laser Doppler velocimeter (LDV) hardware and techniques as well as in a heightened awareness of remaining technological deficiencies. In recent years, significant efforts have been directed toward specifying and reducing the magnitudes of the systematic and random errors contributing to the overall uncertainty of LDV measurements (Ref. 5). One subtle systematic error mechanism not yet widely recognized is that caused by not accounting for or improperly accounting for axis nonorthogonality in multiaxis LDV systems.

This error mechanism was recently encountered in a large transonic wind tunnel application of a two-axis LDV system (Ref. 6). This system was used for free-stream, pitch-plane velocity magnitude, and flow-angle measurements. Orthogonal axis, data reduction equations were initially used; however, the measurement axes were subsequently found to subtend an angle of about 89.1 deg. This oversight resulted in velocity magnitude errors as large as 0.8 percent of measured value and flow-angle measurement errors as large as 0.4 deg. The tunnel test requirements demanded overall uncertainties in velocity magnitude and flow-angle measurements of less than ±1.0 percent of measured value and less than ±0.1 deg. Thus. the oversight errors were obviously unacceptable and were ultimately eliminated by using the equations developed in this report.

The measurement axes of virtually all two- and three-axis LDV systems are either inadvertently or intentionally nonorthogonal to some degree. Also. the LDV axes may be rotated with respect to a reference set of rectangular Cartesian coordinates. Whether or not special data reduction equations are required to account for axis nonorthogonality depends upon the magnitude of the nonorthogonality and the accuracy sought in the particular LDV application. When special equations are necessary. it is essential to incorporate into the transformation relationships the subtle point that the LDV system measures velocity vector projections rather than components when its measurement axes are oblique.

At least two previous analyses have dealt with nonorthogonal LDV measurement axes. The first (Ref. 7) treats a special 3-space case of transmitter-receiver symmetry and is not, therefore, universally applicable. The second (Ref. 8) provides only the 2-space transformation expressions relating two-axis LDV measurements to reference axis components. A general, comprehensive, 3-space analysis of LDV measurement axis nonorthogonality and rotation apparently is not yet available.

The purpose of this report is to derive the general 3-space equations for velocity vector magnitude and spatial orientation in terms of oblique axis parameters and the velocity vector projections measured by each LDV axis. This is accomplished by first employing fundamental tensor relationships to derive expressions that interrelate vector measure numbers, components, and projections. These expressions are then used along with the metric tensor to develop the desired equations.

2.0 VECTOR MEASURE NUMBERS, COMPONENTS, AND PROJECTIONS

Figure 1 depicts a three-dimensional space simultaneously covered by the rectangular Cartesian coordinate system x^i and the general or curvilinear coordinate system $x^{i'}$.
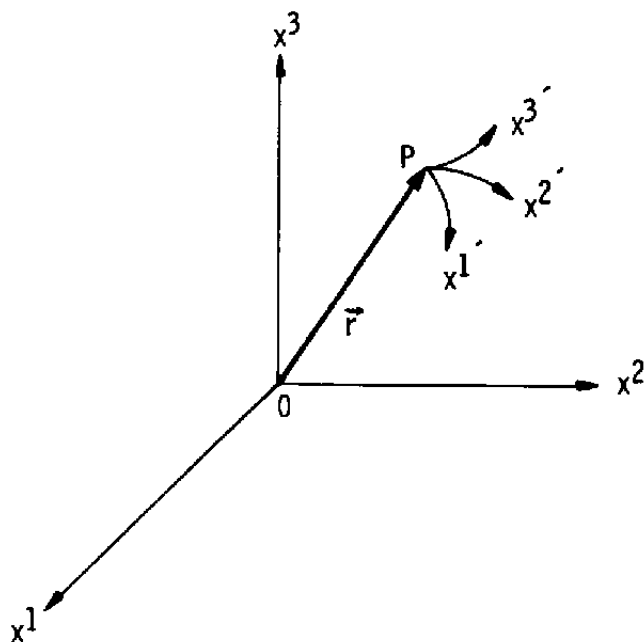
Figure 1. 3-space coordinate systems x^1, x^2, x^3 and $x^{1'}, x^{2'}, x^{3'}$.

Let x^i and $x^{i'}$ be functionally related so that the transformation

$$x^{i'} = x^{i'}(x^i), \quad i' = 1', 2', 3', \text{ and } i = 1, 2, 3 \tag{1}$$

is single valued and reversible. Let the inverse transformation

$$x^i = x^i(x^{i'}), \quad i = 1, 2, 3, \text{ and } i' = 1', 2', 3' \tag{2}$$

also be single valued. Under these conditions the variable point P shown in Fig. 1 can be uniquely specified in terms of either its x^i or $x^{i'}$ coordinates.

Let $\vec{r}$ denote the position vector of the point P with respect to 0, the origin of the rectangular Cartesian coordinate system. A differential change in $\vec{r}$ can be expressed by

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x^{i'}} dx^{i'} \tag{3}$$

where the Einstein summation convention is used.* For unit displacement along the $x^{i'}$ axis, the change in $\vec{r}$ is tangential to $x^{i'}$ and is equal to $\partial\vec{r}/\partial x^{i'}$. The vectors

$$\vec{a}_{i'} = \frac{\partial \vec{r}}{\partial x^{i'}} \tag{4}$$

are the unitary vectors with respect to the point P (Ref. 9) which, in general, constitute a nonorthonormal basis for all field vectors defined at the point P.

The unitary set defines a parallelepiped whose volume is given by the scalar triple product of these vectors (Ref. 9)

$$\mho = \vec{a}_{1'} \cdot \vec{a}_{2'} \times \vec{a}_{3'} \tag{5}$$

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*Summation over the index for all admissible values is to be carried out whenever the same index occurs both as a superscript and as a subscript in a single term. A superscript in the denominator of a term is regarded as a subscript on the quotient. The present analysis is limited to 3-space so that the admissible values are always, 1, 2, and 3.

A second, generally nonorthonormal, set of basis vectors can be defined in terms of the unitary set and their scalar triple product (Ref. 9)

$$\vec{a}^{1\prime} = \frac{1}{\mho} \left(\vec{a}_{2\prime} \times \vec{a}_{3\prime} \right) , \quad \vec{a}^{2\prime} = \frac{1}{\mho} \left(\vec{a}_{3\prime} \times \vec{a}_{1\prime} \right) , \quad \vec{a}^{3\prime} = \frac{1}{\mho} \left(\vec{a}_{1\prime} \times \vec{a}_{2\prime} \right) \tag{6}$$

These are the reciprocal unitary vectors. The unitary and reciprocal unitary vectors are seen to satisfy the condition

$$\vec{a}^{j\prime} \cdot \vec{a}_{i\prime} = \delta^{j\prime}_{i\prime} \tag{7}$$

where $\delta^{j\prime}_{i\prime}$ is the Kronecker delta.*

The differential change in the vector $\vec{r}$ can now be expressed either in terms of the unitary basis as

$$\vec{dr} = \vec{a}_{i\prime} dx^{i\prime} \tag{8}$$

or in terms of the reciprocal unitary basis as

$$\vec{dr} = \vec{a}^{j\prime} dx_{j\prime} \tag{9}$$

Analogously, any field vector $\vec{V}$ defined at the point P can be resolved into components either with respect to the unitary basis as

$$\vec{V} = \vec{a}_{i\prime} v^{i\prime} \tag{10}$$

or with respect to the reciprocal unitary basis as

$$\vec{V} = \vec{a}^{j\prime} v_{j\prime} \tag{11}$$

where $v^{i\prime}$ are the contravariant measure numbers of $\vec{V}$ and $v_{j\prime}$ are the covariant measure numbers. Equations (10) and (11) are statements of the familiar parallelogram law for formation of the vector $\vec{V}$ by vector addition of its components.

The squared length of $\vec{dr}$ is expressed with respect to the unitary set by

$$(ds\prime)^2 = \vec{dr} \cdot \vec{dr} = \vec{a}_{i\prime} \cdot \vec{a}_{j\prime} dx^{i\prime} dx^{j\prime} = g_{i\prime j\prime} dx^{i\prime} dx^{j\prime} \tag{12}$$

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* $\delta^{j\prime}_{i\prime} = 1, \quad i\prime = j\prime$

$\delta^{j\prime}_{i\prime} = 0, \quad i\prime \neq j\prime$

where $g_{i'j'}$ is a symmetric tensor termed the covariant metric tensor (Ref. 10). This same length is expressed with respect to the reciprocal unitary set by

$$(ds')^2 = \vec{dr} \cdot \vec{dr} = \vec{a}^{i'} \cdot \vec{a}^{j'} dx_{i'} \cdot dx_{j'} = g^{i'j'} dx_{i'} dx_{j'} \tag{13}$$

where $g^{i'j'}$ is a symmetric tensor termed the contravariant metric tensor. The analogous expressions for a general field vector $\vec{V}$ are

$$V^2 = g_{i'j'} v^{i'} v^{j'} \tag{14}$$

and

$$V^2 = g^{i'j'} v_{i'} v_{j'} \tag{15}$$

A set of unit vectors collinear with the unitary set is given by

$$\vec{u}_{i'} = \frac{\vec{a}_{i'}}{\sqrt{g_{i'i'}}} \qquad \text{(no summation on } i')\tag{16}$$

The magnitudes of the components of $\vec{V}$ with respect to this unit set are given by

$$(V)_{i'} = v^{i'}\vec{a}_{i'} \cdot \frac{\vec{a}_{i'}}{\sqrt{g_{i'i'}}} = \sqrt{g_{i'i'}} \; v^{i'} \tag{17}$$

The orthogonal projections of $\vec{V}$ onto the axes defined by the unit set are given by

$$(V_\perp)_{i'} = v_{j'}\vec{a}^{j'} \cdot \frac{\vec{a}_{i'}}{\sqrt{g_{i'i'}}} = \frac{v_{i'}}{\sqrt{g_{i'i'}}} = \frac{g_{i'j'} v^{j'}}{\sqrt{g_{i'i'}}} \qquad \text{(no summation on } i')\tag{18}$$

Study of Eqs. (5) and (6) shows that the $a_{i'}$ and $a^{i'}$ are identical quantities when the $a_{i'}$ are the orthonormal basis vectors of a rectangular Cartesian coordinate system. Under these conditions

$$g_{i'j'} = \delta_{i'j'} = g^{i'j'} = \delta^{i'j'} \tag{19}$$

and consequently

$$v_{i'} = v^{i'} = (V)_{i'} = (V_\perp)_{i'} \tag{20}$$

However, $v_{i'}$, $v^{i'}$, $(V)_{i'}$, and $(V_\perp)_{i'}$ are distinctly different quantities when the $a_{i'}$ are nonorthonormal. It will be shown subsequently that an LDV system measures the quantity $(V_\perp)_{i'}$.

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In Riemannian geometric space the element of arc length ds′ is given by Eqs. (12 and 13). If there exists a coordinate transformation

$$x^{i'} = x^{i'}(x^i) \tag{21}$$

such that

$$(ds')^2 = (ds)^2 = \delta_{ij}\, dx^i\, dx^j \tag{22}$$

then the Riemannian space is said to be Euclidean (Ref. 11). In Euclidean space, any coordinate system for which the elements of the metric tensor are constants is termed a Cartesian coordinate system (Ref. 11).

All essential metric (quantitative measures of length, angles, etc.) properties of Euclidean space can be completely specified by use of the applicable metric tensor (Ref. 10). Specifically, Euclidean 3-space covered by the oblique Cartesian coordinate system defined by the transmitted beam pattern of a multiaxis LDV system is characterized by a metric tensor with constant elements. Therefore, it is reasonable to seek mathematical relationships for velocity vector magnitude and spatial orientation in terms of the applicable metric tensor and the LDV-measured values of the velocity vector projections.

3.0 LDV MEASUREMENTS AND VELOCITY VECTOR PROJECTIONS

Most present-day LDV transmitters are of the crossed-beam configuration (Ref. 12). This configuration is compatible with the reference-beam receiver as well as the dual-scatter receiver. The expression for the signal frequency (Doppler frequency) produced by one axis of a crossed-beam, reference-beam LDV system is

$$f_D = \frac{1}{2\pi}\vec{V}\cdot\left(\vec{k}_s - \vec{k}_I\right) \tag{23}$$

where $\vec{V}$ is the velocity vector of a light-scattering particle passing through the crossed-beam or probe volume region, $\vec{k}_I$ is the propagation vector of the illuminating laser beam, and $\vec{k}_s$ is the propagation vector of the scattered, Doppler-shifted laser radiation. For nonrelativistic velocities, Eq. (23) can be put into the form

$$\vec{V}\cdot\vec{u}_{i'} = \left(V_\perp\right)_{i'} = \frac{\lambda}{2\sin\theta/2}f_D \tag{24}$$

where λ is the wavelength of the illuminating laser radiation, θ is the angle subtended by the two intersecting laser beams, and $\vec{u}_i'$ is a unit vector of coplanar with $\vec{k}_s$ and $\vec{k}_I$ and perpendicular to the bisector of the angle θ.

The signal frequency produced by one axis of a crossed-beam, dual-scatter type LDV is given by

$$f_D = \frac{1}{2\pi} \vec{V} \cdot \left(\vec{k}_{I2} - \vec{k}_{I1} \right)$$

(25)

where $\vec{k}_{I1}$ is the propagation vector of the first of two crossed beams of illuminating laser radiation and $\vec{k}_{I2}$ is the propagation vector of the second illuminating beam. With θ as the angle subtended by $\vec{k}_{I1}$ and $\vec{k}_{I2}$, Eq. (25) can be put into a form identical to Eq. (24) which states that an LDV system, whether reference-beam or dual-scatter type, simply measures the orthogonal projection of $\vec{V}$ upon the $\vec{u}_1'$ axis.

4.0 THE METRIC TENSOR

As previously stated, a general coordinate transformation is expressed by

$$x^i = x^i \left(x^{i'} \right)$$

(26)

The total differential of x^i is given by

$$dx^i = \frac{\partial x^i}{\partial x^{i'}} dx^{i'}$$

(27)

The covariant metric tensor is defined by substituting Eq. (27) into the expression for scalar invariance of the infinitesimal distance ds in rectangular Cartesian coordinates and ds' in general curvilinear coordinates (Ref. 13):

$$(ds')^2 = g_{i'j'} dx^{i'} dx^{j'} = (ds)^2 = \delta_{ij} dx^i dx^j$$

(28)

From this it can be seen that

$$g_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}$$

(29)

For the special case of Cartesian coordinates, the general transformation given by Eq. (26) simplifies to

$$x^i = A^i_{i'} x^{i'}$$

(30)

where $A^i_{i'}$ is a matrix of constants independent of coordinate position. Examination of Eqs. (29) and (30) shows that the elements of $A^i_{i'}$ must first be determined before the elements of the metric tensor can be computed.

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4.1 COORDINATE TRANSFORMATIONS IN 3-SPACE

In a Cartesian coordinate system a vector can be represented as a directed line segment connecting two points in the space. Such a vector is shown in Fig. 2 as the directed line segment OP and is identified as $\vec{V}$. The familiar parallelogram law for vector
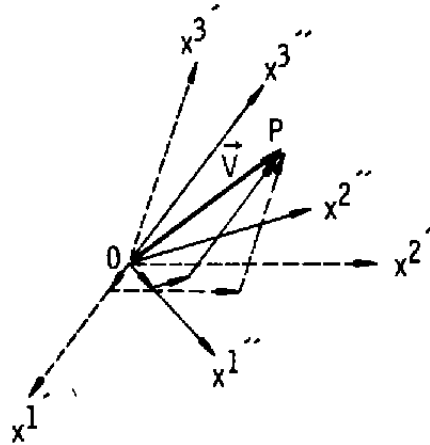
Figure 2. Components of vector $\vec{V}$ with respect to oblique Cartesian coordinate systems $x^{i''}$ and $x^{i'}$.

addition of the vector components with respect to two different oblique Cartesian coordinate systems is illustrated in Fig. 2. Further study of this figure shows that the orthogonal projection of $\vec{V}$ onto the $x^{i''}$ axis is given by

$$x^{1''} + b_{1''2''}x^{2''} + b_{1''3''}x^{3''} = c_{1''1'}x^{1'} + c_{1''2'}x^{2'} + c_{1''3'}x^{3'} \tag{31}$$

where $b_{i''j''}$ is the cosine of the angle subtended by the positive directions of the $x^{i''}$ and $x^{j''}$ axes and $c_{i''j'}$ is the cosine of the angle subtended by the positive directions of the $x^{i''}$ and $x^{i'}$ axes. In general,

$$b_{i''j''}x^{j''} = c_{i''j'}x^{j'} \tag{32}$$

where $b_{i''j''}$ and $c_{i''j'}$ are matrix arrays of these direction cosines. Premultiplication of Eq. (32) by $b^{j''i''}$ yields

$$x^{j''} = b^{j''i''}c_{i''j'}x^{j'} \tag{33}$$

where $b^{j''i''}$ is the inverse of $b_{i''j''}$. A similar procedure produces

$$x^{i'} = b^{i'i'}c_{i'j''}x^{j''} \tag{34}$$

where $b^{j'i'}$ is the inverse of $b_{j'i'}$ and $c_{i'j''}$ is the transpose of $c_{i''j'}$.

For the special case that $x^{j''} = x^j$, where x^j represents a rectangular Cartesian coordinate system, $b^{j''i''}$ becomes the identity matrix I^{ji}

$$x^j = I^{ji}c_{ij'}x^{j'} = A^j_{j'}x^{j'} \tag{35}$$

which is in agreement with Eq. (30). Equation (34) now becomes

$$x^{j'} = b^{i'i'}c_{i'j}x^j = D^{j'}_j x^j \tag{36a}$$

where

$$D^{1'}_1 = \frac{1}{\Delta'}\left\{ c_{1'1}\left[1 - \left(b_{2'3'}\right)^2\right] + c_{2'1}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) + c_{3'1}\left(b_{1'2'}b_{2'3'} - b_{1'3'}\right)\right\} \tag{36b}$$

$$D^{1'}_2 = \frac{1}{\Delta'}\left\{ c_{1'2}\left[1 - \left(b_{2'3'}\right)^2\right] + c_{2'2}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) + c_{3'2}\left(b_{1'2'}b_{2'3'} - b_{1'3'}\right)\right\} \tag{36c}$$

$$D^{1'}_3 = \frac{1}{\Delta'}\left\{ c_{1'3}\left[1 - \left(b_{2'3'}\right)^2\right] + c_{2'3}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) - c_{3'3}\left(b_{1'2'}b_{2'3'} - b_{1'3'}\right)\right\} \tag{36d}$$

$$D^{2'}_1 = \frac{1}{\Delta'}\left\{ c_{1'1}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) - c_{2'1}\left[1 - \left(b_{1'3'}\right)^2\right] + c_{3'1}\left(b_{1'2'}b_{1'3'} - b_{2'3'}\right)\right\} \tag{36e}$$

$$D^{2'}_2 = \frac{1}{\Delta'}\left\{ c_{1'2}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) + c_{2'2}\left[1 - \left(b_{1'3'}\right)^2\right] + c_{3'2}\left(b_{1'2'}b_{1'3'} - b_{2'3'}\right)\right\} \tag{36f}$$

$$D^{2'}_3 = \frac{1}{\Delta'}\left\{ c_{1'3}\left(b_{1'3'}b_{2'3'} - b_{1'2'}\right) + c_{2'3}\left[1 - \left(b_{1'3'}\right)^2\right] + c_{3'3}\left(b_{1'2'}b_{1'3'} - b_{2'3'}\right)\right\} \tag{36g}$$

$$D^{3'}_1 = \frac{1}{\Delta'}\left\{ c_{1'1}\left(b_{1'2'}b_{2'3'} - b_{1'3'}\right) + c_{2'1}\left(b_{1'2'}b_{1'3'} - b_{2'3'}\right) + c_{3'1}\left[1 - \left(b_{1'2'}\right)^2\right]\right\} \tag{36h}$$

$$D_2^{3\prime} = \frac{1}{\Delta'} \left\{ c_{1\,'2} \left(b_{1\,'2\,'} b_{2\,'3\,'} - b_{1\,'3\,'} \right) + c_{2\,'2} \left(b_{1\,'2\,'} b_{1\,'3\,'} - b_{2\,'3\,'} \right) + c_{3\,'2} \left[1 - \left(b_{1\,'2\,'} \right)^2 \right] \right\}$$

$$\tag{36i}$$

$$D_3^{3\prime} = \frac{1}{\Delta'} \left\{ c_{1\,'3} \left(b_{1\,'2\,'} b_{2\,'3\,'} - b_{1\,'3\,'} \right) + c_{2\,'3} \left(b_{1\,'2\,'} b_{1\,'3\,'} - b_{2\,'3\,'} \right) + c_{3\,'3} \left[1 - \left(b_{1\,'2\,'} \right)^2 \right] \right\}$$

$$\tag{36j}$$

$$\Delta' = 1 - \left(b_{1\,'2\,'} \right)^2 - \left(b_{2\,'3\,'} \right)^2 - \left(b_{1\,'3\,'} \right)^2 + 2\, b_{1\,'2\,'} b_{2\,'3\,'} b_{1\,'3\,'}$$

$$\tag{36k}$$

4.2 ELEMENTS OF THE METRIC TENSOR

Expansion of Eq. (35) yields the expressions for the rectangular coordinates as a function of the oblique coordinates. Differentiation of these expressions and substitution into Eq. (29) yield values for the elements of the covariant metric tensor:

$$g_{1\,'1\,'} = \left[\left(c_{11'} \right)^2 + \left(c_{21'} \right)^2 + \left(c_{31'} \right)^2 \right] \tag{37a}$$

$$g_{2\,'2\,'} = \left[\left(c_{12'} \right)^2 + \left(c_{22'} \right)^2 + \left(c_{32'} \right)^2 \right] \tag{37b}$$

$$g_{3\,'3\,'} = \left[\left(c_{13'} \right)^2 + \left(c_{23'} \right)^2 + \left(c_{33'} \right)^2 \right] \tag{37c}$$

$$g_{2\,'1\,'} = g_{1\,'2\,'} = \left[\left(c_{11'} \right) \left(c_{12'} \right) + \left(c_{21'} \right) \left(c_{22'} \right) + \left(c_{31'} \right) \left(c_{32'} \right) \right] \tag{37d}$$

$$g_{3\,'1\,'} = g_{1\,'3\,'} = \left[\left(c_{11'} \right) \left(c_{13'} \right) + \left(c_{21'} \right) \left(c_{23'} \right) + \left(c_{31'} \right) \left(c_{33'} \right) \right] \tag{37e}$$

$$g_{3\,'2\,'} = g_{2\,'3\,'} = \left[\left(c_{12'} \right) \left(c_{13'} \right) + \left(c_{22'} \right) \left(c_{23'} \right) + \left(c_{32'} \right) \left(c_{33'} \right) \right] \tag{37f}$$

5.0 VECTOR MAGNITUDE

Expanding Eq. (14) and taking the square root yield the expression for the magnitude of $\vec{V}$ in terms of the elements of the covariant metric tensor and the contravariant measure numbers:

$$V = \left[g_{1'1'} \left(v^{1'} \right)^2 + g_{2'2'} \left(v^{2'} \right)^2 + g_{3'3'} \left(v^{3'} \right)^2 - 2g_{1'2'} v^{1'} v^{2'} \right.$$
$$\left. + 2 g_{1'3'} v^{1'} v^{3'} + 2 g_{2'3'} v^{2'} v^{3'} \right]^{1/2} \tag{38}$$

Solution of Eq. (17) for the $v^{i'}$ and substitution into Eq. (38) give the expression for vector magnitude in terms of the vector components:

$$V = \left[(V)_1^2 + (V)_2^2 + (V)_3^2 + 2 g_{1'2'} \left(g_{1'1'} g_{2'2'} \right)^{-1/2} (V)_1 \cdot (V)_2 \right.$$
$$\left. + 2 g_{1'3'} \left(g_{1'1'} g_{3'3'} \right)^{-1/2} (V)_1 \cdot (V)_3 + 2 g_{2'3'} \left(g_{2'2'} g_{3'3'} \right)^{-1/2} (V)_2 \cdot (V)_3 \right]^{1/2} \tag{39}$$

The general expression for vector magnitude in terms of vector projections is quite complicated. For practical applications, however, a straightforward procedure is to solve Eq. (18) for the $v^{j'}$ in terms of the $(V_\perp)_{i'}$ and to then substitute the $v^{j'}$ into Eq. (38). The $v^{j'}$ are given by

$$v^{1'} = \frac{1}{\Delta} \left\{ \left(g_{1'1'} \right)^{1/2} \left(V_\perp \right)_{1'} \left[g_{2'2'} g_{3'3'} - \left(g_{2'3'} \right)^2 \right] \right.$$
$$- \left(g_{2'2'} \right)^{1/2} \left(V_\perp \right)_{2'} \left[g_{1'2'} g_{3'3'} - g_{1'3'} g_{2'3'} \right]$$
$$\left. + \left(g_{3'3'} \right)^{1/2} \left(V_\perp \right)_{3'} \left[g_{1'2'} g_{2'3'} - g_{1'3'} g_{2'2'} \right] \right\} \tag{40a}$$

$$v^{2'} = \frac{1}{\Delta} \left\{ - \left(g_{1'1'} \right)^{1/2} \left(V_\perp \right)_{1'} \left[g_{1'2'} g_{3'3'} - g_{1'3'} g_{2'3'} \right] \right.$$
$$+ \left(g_{2'2'} \right)^{1/2} \left(V_\perp \right)_{2'} \left[g_{1'1'} g_{3'3'} - \left(g_{1'3'} \right)^2 \right]$$
$$\left. - \left(g_{3'3'} \right)^{1/2} \left(V_\perp \right)_{3'} \left[g_{1'1'} g_{2'3'} - g_{1'2'} g_{1'3'} \right] \right\} \tag{40b}$$

$$v^{3'} = \frac{1}{\Delta} \left\{ \left(g_{1'1'} \right)^{1/2} \left(V_\perp \right)_{1'} \left[g_{1'2'} g_{2'3'} - g_{1'3'} g_{2'2'} \right] \right.$$
$$- \left(g_{2'2'} \right)^{1/2} \left(V_\perp \right)_{2'} \left[g_{1'1'} g_{2'3'} - g_{1'2'} g_{1'3'} \right]$$
$$\left. - \left(g_{3'3'} \right)^{1/2} \left(V_\perp \right)_{3'} \left[g_{1'1'} g_{2'2'} - \left(g_{1'2'} \right)^2 \right] \right\} \tag{40c}$$

where

$$\Delta = g_{1'1'}\left[g_{2'2'}g_{3'3'} - \left(g_{2'3'} \right)^2 \right] - g_{1'2'}\left[g_{1'2'}g_{3'3'} - g_{1'3'}g_{2'3'} \right]$$
$$+ g_{1'3'}\left[g_{1'2'} - g_{2'2'}g_{1'3'} \right] \qquad (40d)$$

6.0 ANGLE SUBTENDED BY TWO INTERSECTING VECTORS

Dividing Eq. (14) by V^2 yields

$$1 = g_{i'j'}\frac{v^{i'}}{V}\frac{v^{j'}}{V} \qquad (41)$$

so that $v^{i'}/V$ is a measure number of the unit vector $\vec{u}_V$. Assume that a vector $\vec{W}$ intersects $\vec{V}$ to define the angle $\phi_{V,W}$ as shown in Fig. 3. The unit vector $\vec{u}_w$ and unit
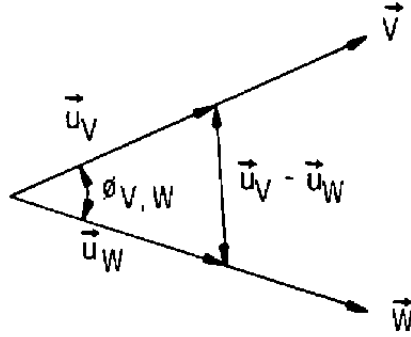
Figure 3. Angle $\phi_{V,W}$ subtended by the intersection of vectors $\vec{V}$ and $\vec{W}$.

vector measure number $W^{i'}/W$ are associated with the vector $\vec{W}$. The magnitude of the vector $(\vec{u}_V - \vec{u}_w)$ can be found by the cosine law to be

$$\left| \vec{u}_V - \vec{u}_W \right|^2 = 2\left(1 - \cos\phi_{V,W} \right) \qquad (42)$$

This same vector magnitude is given by [see Eq. (24)]

$$\left| \vec{u}_V - \vec{u}_W \right|^2 = g_{i'j'}\left(\frac{v^{i'}}{V} - \frac{w^{i'}}{W} \right)\left(\frac{v^{j'}}{V} - \frac{w^{j'}}{W} \right)$$
$$= g_{i'j'}\frac{v^{i'}}{V}\frac{v^{j'}}{V} + g_{i'j'}\frac{w^{i'}}{W}\frac{w^{j'}}{W} - 2g_{i'j'}\frac{v^{i'}}{V}\frac{w^{i'}}{W}$$
$$= 2\left(1 - g_{i'j'}\frac{v^{i'}}{V}\frac{w^{j'}}{W} \right) \qquad (43)$$

Comparison of Eqs. (42) and (43) shows that

$$\phi_{V,W} = \arccos \left(g_{i'j'} \, \frac{v^{i'}}{V} \, \frac{w^{j'}}{W} \right)$$

(44)

6.1 ANGLE SUBTENDED BY $\vec{V}$ AND A RECTANGULAR COORDINATE AXIS

The set of measure numbers of a vector is a univalent (rank one) tensor (Ref. 13.) A contravariant, univalent tensor is defined as a quantity that transforms as (Ref. 13)

$$u^{i'} = \frac{\partial x^{i'}}{\partial x^i} u^i$$

(45)

Once again let x^i represent a rectangular Cartesian coordinate system, and let $x^{i'}$ represent an oblique Cartesian coordinate system. A unit vector $\vec{u_i}$ in the x^1 coordinate direction has a single non-zero measure number $u^1 = 1$. The corresponding measure numbers in the oblique coordinate system are given by

$$u^{i'} = \frac{\partial x^{i'}}{\partial x^1} u^1 \left(u^1 = 1, u^j = u^k = 0 \right)$$

(46)

Now Eq. (44) can be used to find the angle subtended by the vector $\vec{V}$ and the x^i coordinate axis.

$$\phi_{V,u} = \arccos \left(g_{i'j'} \, \frac{v^{i'} u^{j'}}{V} \right)$$

(47)

At this point Eqs. (36) and (45) can be used to find the $u^{i'}$ in the terms of u^1. The $u^{i'}$ values can then be substituted into Eq. (47) to obtain

$$\phi_{V,x^1} = \arccos \left\{ \frac{1}{V} \left[g_{1'1'} D_1^{1'} v^{1'} + g_{2'2'} D_1^{2'} v^{2'} + g_{3'3'} D_1^{3'} v^{3'} \right. \right.$$
$$+ g_{1'2'} D_1^{2'} v^{1'} + g_{2'1'} D_1^{1'} v^{2'} + g_{1'3'} D_1^{3'} v^{1'}$$
$$\left. \left. + g_{3'1'} D_1^{1'} v^{3'} + g_{2'3'} D_1^{3'} v^{2'} + g_{3'2'} D_1^{2'} v^{3'} \right] \right\}$$

(48)

The expressions for the angle ϕ in terms of vector components can be found by substituting Eq. (17) into Eq. (48):

$$\phi_{V,x^1} = \arccos\left\{\frac{1}{V}\left\{\left[D_1^{1'}g_{1'1'} + D_1^{2'}g_{1'2'} + D_1^{3'}g_{1'3'}\right]\frac{(V)_{1'}}{(g_{1'1'})^{1/2}}\right.\right.$$

$$+ \left[D_1^{1'}g_{2'1'} + D_1^{2'}g_{2'2'} + D_1^{3'}g_{2'3'}\right]\frac{(V)_{2'}}{(g_{2'2'})^{1/2}}$$

$$\left.\left.+ \left[D_1^{1'}g_{3'1'} - D_1^{2'}g_{3'2'} + D_1^{3'}g_{3'3'}\right]\frac{(V)_{3'}}{(g_{3'3'})^{1/2}}\right\}\right\} \tag{49a}$$

$$\phi_{V,x^2} = \arccos\left\{\frac{1}{V}\left\{\left[D_2^{1'}g_{1'1'} + D_2^{2'}g_{1'2'} + D_2^{3'}g_{1'3'}\right]\frac{(V)_{1'}}{(g_{1'1'})^{1/2}}\right.\right.$$

$$+ \left[D_2^{1'}g_{2'1'} + D_2^{2'}g_{2'2'} + D_2^{3'}g_{2'3'}\right]\frac{(V)_{2'}}{(g_{2'2'})^{1/2}}$$

$$\left.\left.+ \left[D_2^{1'}g_{3'1'} + D_2^{2'}g_{3'2'} + D_2^{3'}g_{3'3'}\right]\frac{(V)_{3'}}{(g_{3'3'})^{1/2}}\right\}\right\} \tag{49b}$$

$$\phi_{V,x^3} = \arccos\left\{\frac{1}{V}\left\{\left[D_3^{1'}g_{1'1'} + D_3^{2'}g_{1'2'} + D_3^{3'}g_{1'3'}\right]\frac{(V)_{1'}}{(g_{1'1'})^{1/2}}\right.\right.$$

$$+ \left[D_3^{1'}g_{2'1'} + D_3^{2'}g_{2'2'} + D_3^{3'}g_{2'3'}\right]\frac{(V)_{2'}}{(g_{2'2'})^{1/2}}$$

$$\left.\left.+ \left[D_3^{1'}g_{3'1'} + D_3^{2'}g_{3'2'} + D_3^{3'}g_{3'3'}\right]\frac{(V)_{3'}}{(g_{3'3'})^{1/2}}\right\}\right\} \tag{49c}$$

The general, 3-space expressions for ϕ in terms of the $(V_\perp)_{i'}$ projections are quite unwieldy. For practical applications, the measured values of the $(V_\perp)_{i'}$ can be inserted into Eqs. (40 a through c); then the resultant $v^{i'}$ numerical values can be substituted into Eq. (48) to find ϕ.

7.0 COMPONENT AND PROJECTION INTERRELATIONSHIPS

For purposes of analysis it is often convenient to use expressions that interrelate components and projections in the x^i and $x^{i'}$ coordinate systems. The familiar parallelogram law for vector addition of components can be used to describe the vector quantity $\vec{V}$ in the x^i as well as the $x^{i'}$ coordinate systems (see Fig. 2):

$$\vec{V} = \sum_{i=1}^{3} (V)_i \, \vec{u}_i \tag{50}$$

and

$$\vec{V} = \sum_{i=1}^{3'} (V)_{i'} \vec{u}_{i'} \tag{51}$$

where $\vec{u}i$ and u_i' are unit vectors collinear with the x^i and $x^{i'}$ axes respectively.

By forming the scalar product of $\vec{V}$ as expressed in Eq. (50) with $\vec{u}_j$, one finds that

$$\begin{bmatrix} (V_\perp)_1 \\ (V_\perp)_2 \\ (V_\perp)_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (V)_1 \\ (V)_2 \\ (V)_3 \end{bmatrix} \tag{52}$$

which simply formalizes the earlier statement that $(V_\perp)_i$ and $(V)_i$ are identical quantities in the rectangular Cartesian coordinate system x^i. Forming the scalar product of $\vec{V}$ as expressed in Eq. (51) with $\vec{u}_i'$ yields the relationships for $(V_\perp)_{i'}$ in terms of $(V)_{i'}$:

$$\begin{bmatrix} (V_\perp)_{1'} \\ (V_\perp)_{2'} \\ (V_\perp)_{3'} \end{bmatrix} = \begin{bmatrix} 1 & b_{1'2'} & b_{1'3'} \\ b_{2'1'} & 1 & b_{2'3'} \\ b_{3'1'} & b_{3'2'} & 1 \end{bmatrix} \begin{bmatrix} (V)_{1'} \\ (V)_{2'} \\ (V)_{3'} \end{bmatrix} \tag{53}$$

The x^i components in terms of $x^{i'}$ components can be found by forming the scalar product of Eq. (50) and Eq. (51) with $\vec{u}_i$ and equating the results:

$$\begin{bmatrix} (V)_1 \\ (V)_2 \\ (V)_3 \end{bmatrix} = \begin{bmatrix} c_{11'} & c_{12'} & c_{13'} \\ c_{21'} & c_{22'} & c_{23'} \\ c_{31'} & c_{32'} & c_{33'} \end{bmatrix} \begin{bmatrix} (V)_{1'} \\ (V)_{2'} \\ (V)_{3'} \end{bmatrix} \tag{54}$$

The functional interrelationship of $(V_\perp)_{i'}$ and $(V)_{i'}$ can be found by forming the scalar product of Eq. (50) and Eq. (51) with $\vec{u}_i'$ and equating the results:

$$
\begin{bmatrix} (V_{\perp})_{1'} \\ (V_{\perp})_{2'} \\ (V_{\perp})_{3'} \end{bmatrix} = \begin{bmatrix} c_{1'1} & c_{1'2} & c_{1'3} \\ c_{2'1} & c_{2'2} & c_{2'3} \\ c_{3'1} & c_{3'2} & c_{3'3} \end{bmatrix} \begin{bmatrix} (V)_1 \\ (V)_2 \\ (V)_3 \end{bmatrix}
\tag{55}
$$

The inverse relationships for Eqs. (52 through 55) can readily be found. Because of its usefulness, however, the inverse relationship for Eq. (55) is given here:

$$
\begin{bmatrix} (V)_1 \\ (V)_2 \\ (V)_3 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} (c_{2'2}c_{3'3} - c_{3'2}c_{2'3}) & (c_{3'2}c_{1'3} - c_{1'2}c_{3'3}) & (c_{1'2}c_{2'3} - c_{2'2}c_{1'3}) \\ (c_{3'1}c_{2'3} - c_{2'1}c_{3'3}) & (c_{1'1}c_{3'3} - c_{3'1}c_{1'3}) & (c_{2'1}c_{1'3} - c_{1'1}c_{2'3}) \\ (c_{2'1}c_{3'2} - c_{3'1}c_{2'2}) & (c_{3'1}c_{1'2} - c_{1'1}c_{3'2}) & (c_{1'1}c_{2'2} - c_{2'1}c_{1'2}) \end{bmatrix} \begin{bmatrix} (V_{\perp})_{1'} \\ (V_{\perp})_{2'} \\ (V_{\perp})_{3'} \end{bmatrix}
\tag{56}
$$

where

$$
\Delta = c_{1'1}(c_{2'2}c_{3'3} - c_{2'3}c_{3'2}) + c_{1'2}(c_{3'1}c_{2'3} - c_{2'1}c_{3'3}) + c_{1'3}(c_{2'1}c_{3'2} - c_{3'1}c_{2'2})
$$

8.0 2-SPACE RELATIONSHIPS

The 2-space versions of the preceding relationships are important because of the widespread use of two-axis LDV systems for two-dimensional flow studies. For simplification let x^3 be collinear with $x^{3'}$, and let both be perpendicular to $x^{1'}, x^{2'}, x^1$ and x^2. Then, for 2-space we have

$$
v^{3'} = (V)_{3'} = (V_{\perp})_{3'} = c_{31'} = c_{32'} = c_{3'1} = c_{3'2}
$$

$$
= D_3^{1'} = D_3^{2'} = D_1^{3'} = D_2^{3'} = D_3^{3'} = g_{1'3'} = g_{2'3'} = 0
\tag{57}
$$

It can also be readily shown that

$$
c_{3'3} = c_{33'} = g_{1'1'} = g_{2'2'} = 1
\tag{58}
$$

$$
g_{1'2'} = b_{1'2'}
\tag{59}
$$

$$
D_1^{1'} = \frac{1}{\sin^2\gamma} \left[c_{1'1} - c_{2'1} b_{1'2'} \right]
\tag{60}
$$

$$D_2^{1'} = \frac{1}{\sin^2\gamma}\left[c_{1'2} - c_{2'2}\,b_{1'2'}\right] \tag{61}$$

$$D_1^{2'} = \frac{1}{\sin^2\gamma}\left[-c_{1'1}\,b_{1'2'} + c_{2'1}\right] \tag{62}$$

$$D_2^{2'} = \frac{1}{\sin^2\gamma}\left[-c_{1'2}\,b_{1'2'} + c_{2'2}\right] \tag{63}$$

where $\gamma = \arccos b_{1'2'}$. $\tag{64}$

8.1 VECTOR MAGNITUDE

In 2-space, Eq. (38) becomes

$$V = \left[\left(v^{1'}\right)^2 + \left(v^{2'}\right)^2 - 2\,b_{1'2'}\,v^{1'}v^{2'}\right]^{\frac{1}{2}} \tag{65}$$

while the expression for V in terms of vector components, Eq. (39), reduces to

$$V = \left[(V)_{1'}^2 + (V)_{2'}^2 + 2\,b_{1'2'}\cdot(V)_{1'}\cdot(V)_{2'}\right]^{\frac{1}{2}} \tag{66}$$

Substitution of Eqs. (58) and (59) into Eqs. (40a), (40b), and (40d) yields

$$v^{1''} = \frac{1}{\sin^2\gamma}\left[\left(V_\perp\right)_{1'} - b_{1'2'}\left(V_\perp\right)_{2'}\right] \tag{67}$$

$$v^{2'} = \frac{1}{\sin^2\gamma}\left[-b_{1'2'}\left(V_\perp\right)_{1'} + \left(V_\perp\right)_{2'}\right] \tag{68}$$

Now Eqs. (67) and (68) can be substituted into Eq. (65) to find the simplified 2-space expression for V in terms of vector projections:

$$V = \left\{\frac{1}{\sin^2\gamma}\left[\left(V_\perp\right)_{1'}^2 - \left(V_\perp\right)_{2'}^2 - 2\,b_{1'2'}\left(V_\perp\right)_{1'}\cdot\left(V_\perp\right)_{2'}\right]\right\}^{\frac{1}{2}} \tag{69}$$

It can be seen that Eqs. (66) and (69) both reduce to the familiar form for rectangular Cartesian coordinates when $\gamma = 90$ deg·

$$V = \left\{(v)_{1'}^2 - (v_{2'}^2\right\}^{\frac{1}{2}} \tag{70}$$

8.2 THE ANGLE ϕ

In 2-space, the simplified expressions for ϕ in terms of vector components are found to be

$$\phi_{V,x}{}^1 = \arccos\left\{\frac{1}{V}\left[c_{1'1}(V)_{1'} + c_{2'1}(V)_{2'}\right]\right\} \tag{71}$$

$$\phi_{V,x}{}^2 = \arccos\left\{\frac{1}{V}\left[c_{1'2}(V)_{1'} + c_{2'2}(V)_{2'}\right]\right\} . \tag{72}$$

The expressions for ϕ in terms of velocity projections are given by

$$\phi_{V,x}{}^1 = \arccos\left\{\frac{1}{V\sin^2\gamma}\left[(c_{1'1} - c_{2'1}b_{1'2'})(V_\perp)_{1'} + (c_{2'1} - c_{1'1}b_{1'2'})(V_\perp)_{2'}\right]\right\} \tag{73}$$

and

$$\phi_{V,x}{}^2 = \arccos\left\{\frac{1}{V\sin^2\gamma}\left[(c_{1'2} - c_{2'2}b_{1'2'})(V_\perp)_{1'} + (c_{2'2} - c_{1'2}b_{1'2'})(V_\perp)_{2'}\right]\right\} \tag{74}$$

It can be seen that Eqs. (71) and (73) and Eqs. (72) and (74) reduce to the familiar forms for rectangular Cartesian coordinates when $\gamma = 90$ deg:

$$\phi_{V,x}{}^1 = \arccos\left\{\frac{1}{V}\left[c_{1'1}(V)_{1'} + c_{2'1}(V)_{2'}\right.\right. \tag{75}$$

and

$$\phi_{V,x}{}^1 = \arccos\left\{\frac{1}{V}\left[c_{1'2}(V)_{1'} + c_{2'2}(V)_{2'}\right]\right\} \tag{76}$$

8.3 COMPONENTS AND PROJECTIONS

In 2-space, Eqs. (52) through (56) reduce to

$$\begin{bmatrix} (V_\perp)_1 \\ (V_\perp)_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (V)_1 \\ (V)_2 \end{bmatrix} \tag{77}$$

$$
\begin{bmatrix} (V_\perp)_{1'} \\ (V_\perp)_{2'} \end{bmatrix} = \begin{bmatrix} 1 & b_{1'2'} \\ b_{2'1'} & 1 \end{bmatrix} \begin{bmatrix} (V)_{1'} \\ (V)_{2'} \end{bmatrix} \tag{78}
$$

$$
\begin{bmatrix} (V)_1 \\ (V)_2 \end{bmatrix} = \begin{bmatrix} c_{11'} & c_{12'} \\ c_{21'} & c_{22'} \end{bmatrix} \begin{bmatrix} (V)_{1'} \\ (V)_{2'} \end{bmatrix} \tag{79}
$$

$$
\begin{bmatrix} (V_\perp)_{1'} \\ (V_\perp)_{2'} \end{bmatrix} = \begin{bmatrix} c_{1'1} & c_{1'2} \\ c_{2'1} & c_{2'2} \end{bmatrix} \begin{bmatrix} (V)_1 \\ (V)_2 \end{bmatrix} \tag{80}
$$

$$
\begin{bmatrix} (V)_1 \\ (V)_2 \end{bmatrix} = \frac{1}{\sin\gamma} \begin{bmatrix} c_{2'2} & -c_{1'2} \\ -c_{2'1} & c_{1'1} \end{bmatrix} \begin{bmatrix} (V_\perp)_{1'} \\ (V_\perp)_{2'} \end{bmatrix} \tag{81}
$$

9.0 CONCLUSIONS

The complexity of the general 3-space relationships clearly illustrates the desirability of implementing orthogonal LDV measurement axes collinear with the reference system axes. When these conditions are unattainable, the equations in this report can be used to compute the values for velocity magnitude and spatial orientation within the uncertainties dictated by the remaining LDV system error mechanisms.

The provision of the correct equations to account for axis nonorthogonality and rotation now focuses attention upon the status, with regard to accuracy and precision, of techniques and devices for determining the $b_{i'j'}$ and $c_{i'j}$ values to be used in these equations. Moreover, further reductions in LDV instrument errors will require additional refinement of (1) techniques and devices for determining the signal frequency-to-velocity calibration factor for each LDV measurement axis and (2) techniques and devices for measuring the signal frequency associated with each axis.

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NOMENCLATURE

$A^i_{i'}$	Matrix array of constants, Eq. (30)
$\vec{a}_{i'}$	Unitary vector, Eq. (4)
$\vec{a}^{i'}$	Reciprocal unitary vector, Eq. (6)
$b_{j'_1'}$	Matrix array of direction cosines, Eq. (34)
$b^{j'}_{1'}$	Inverse of $b_{j'_1'}$, Eq. (36a)
$c_{1'j}$	Matrix array of direction cosines, Eq. (36a)
$D^{j'}_j$	Matrix array of constants, Eq. (36a)
$\vec{dr}$	Differential displacement vector, Eq. (3)
ds	Differential displacement magnitude in the x^1 coordinate system, Eq. (22)
ds'	Differential displacement magnitude in the $x^{i'}$ coordinate system, Eq. (12)
f_D	LDV signal frequency, Eq. (23)
$g_{i'j'}$	Covariant metric tensor, Eq. (12)
$g^{i'j'}$	Contravariant metric tensor, Eq. (13)
$\vec{k}_s$	Scatter radiation propagation vector, Eq. (23)
$\vec{k}_1$	Illuminating beam propagation vector, Eqs. (23) and (25)
$\vec{r}$	Position vector, Eq. (3)
$\vec{u}_i$	Unit vector collinear with the x^i axis, Eq. (50)
$\vec{u}_{i'}$	Unit vector collinear with the $x^{i'}$ axis, Eq. (24)
$\vec{u}_v$	Unit vector collinear with $\vec{V}$, Eq. (42)
$\vec{u}_w$	Unit vector collinear with $\vec{W}$, Eq. (42)
$\mho$	Volume, Eq. (5)
$\vec{V}$	Vector, Eq. (10)

V   Magnitude of $\vec{V}$, Eq. (14)

$(V)_i$ Component of $\vec{V}$ collinear with the x^i axis, Eq. (52)

$(V_\perp)_i$ Orthogonal projection of $\vec{V}$ collinear with the x^i axis, Eq. (52)

$(V)_{i'}$ Component of $\vec{V}$ collinear with the $x^{i'}$ axis, Eq. (17)

$(V_\perp)_{i'}$ Orthogonal projection of $\vec{V}$ collinear with the $x^{i'}$ axis, Eq. (18)

$v_{j'}$ Covariant measure number of the vector $\vec{V}$, Eq. (11)

$v^{i'}$ Contravariant measure number of the vector $\vec{V}$, Eq. (10)

$\vec{W}$ Vector, Eq. (42)

W   Magnitude of $\vec{W}$, Eq. (43)

x^i Rectangular Cartesian coordinates, Eq. (2)

$x^{i'}$ Oblique Cartesian coordinates, Eq. (1)

$x^{i''}$ Oblique Cartesian coordinates, Eq. (31)

γ Angle equal to arccos $b_{1'2'}$, Eqs. (60) through (64)

Δ A constant, Eq. (56)

Δ' A constant, Eqs. (36b) through (36k)

δ Kronecker delta, Eqs. (7) and (19)

θ Angle subtended by LDV transmitter beams, Eq. (24)

λ Wavelength of laser radiation, Eq. (24)

$\phi_{V,W}$ Angle subtended by the intersection of $\vec{V}$ and $\vec{W}$, Eq. (42)

ϕ_{V,x^i} Angle subtended by the intersection of $\vec{V}$ and x^i coordinate axis, Eq. (48)